

**A remark to “A note on the Fundamental Group of a
Triangular Algebra”, by F. Xu.**

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ABSTRACT. We provide a counter-example to Proposition 3.2 of [4].

In [1], the quiver Γ of homotopy relations of admissible presentations of an algebra (as quotients of some path algebra) was introduced. In [3], P. Le Meur showed that if $A \simeq kQ/I$ and

- I is a monomial ideal and
- the quiver Q has no multiple arrows

then the quiver Γ has a unique source, without any additional hypothesis about the characteristic of the field k .

Furthermore, in [2], P. Le Meur showed that if

- k is a field of characteristic zero, and
- the quiver Q has no double bypasses,

then the quiver Γ has a unique source.

These results ensure that under some hypotheses, there is a privileged homotopy relation, and consequently, a privileged fundamental group among all the fundamental groups that can arise as fundamental groups of some presentation of a given algebra.

In [2], Example 3, page 345, shows that the second quoted result is not true if one drops the two hypotheses simultaneously. In that example, which consists of a quiver having a double bypass, and considering a field of characteristic 2 one obtains a quiver Γ with two sources. Moreover, there is a suggested generalization of this example to any non-zero value of $\text{char } k$.

It is then natural to ask if one can drop one of the two hypotheses to generalize Le Meur’s result. The question has been tackled in [4]. In that paper, the framework is that of triangular algebras over fields of characteristic zero. One can find the following Proposition:

Proposition 3.2 [4] *Assume the underlying quiver contains no oriented cycles (and k is a field of characteristic zero). Then Γ has a unique source.*

The following counter-example shows that this result is not true.

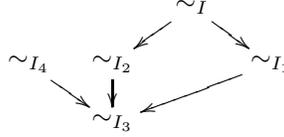
Recall from [2] that given a bypass (α, u) in a quiver Q , then $\phi_{\alpha, u, \tau}$ denotes the automorphism of kQ which sends α to $\alpha + \tau u$, $\tau \in k \setminus \{0\}$, and leaves the other arrows fixed.

Key words and phrases. Fundamental group, bound quiver, triangular algebra.

Counter-example: Let k be a field of characteristic not equal 2, and $A = kQ/I$, where Q is the quiver $3 \begin{array}{c} \xrightarrow{\beta_1} \\ \xleftarrow{\beta_1} \end{array} 2 \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\alpha_2} \end{array} 1$, and $I = \langle \alpha_1\beta_1, \alpha_2\beta_2 \rangle$. Since I is monomial, the fundamental group $\pi_1(Q, I)$ is isomorphic to the free group in two generators, $\mathbb{Z} \amalg \mathbb{Z}$. A straightforward computation shows that $kQ/I \simeq kQ/I_1 \simeq kQ/I_2 \simeq kQ/I_3 \simeq kQ/I_4$, where:

- $I_1 = \langle (\alpha_1 - \alpha_2)\beta_1, \alpha_2\beta_2 \rangle = \phi_{\alpha_1, \alpha_2, -1}(I)$, and leads to a fundamental group isomorphic to \mathbb{Z} ;
- $I_2 = \langle \alpha_1(\beta_1 - \beta_2), \alpha_2\beta_2 \rangle = \phi_{\beta_1, \beta_2, -1}(I)$, and leads to a fundamental group isomorphic to \mathbb{Z} ;
- $I_3 = \langle \alpha_1\beta_1 - \alpha_2\beta_1 - \alpha_1\beta_2, \alpha_2\beta_2 \rangle = \phi_{\alpha_1, \alpha_2, -1}(I_2) = \phi_{\beta_1, \beta_2, -1}(I_1)$, and leads to a trivial fundamental group;
- $I_4 = \langle \alpha_1\beta_1 + \alpha_2\beta_2, \alpha_2\beta_1 + \alpha_1\beta_2 \rangle$. The ideal I_4 leads to a fundamental group isomorphic to \mathbb{Z}_2 and is obtained from the automorphism ϕ of kQ defined by $\phi(\alpha_1) = \frac{1}{2}(\alpha_1 - \alpha_2)$, $\phi(\alpha_2) = \frac{1}{2}(\alpha_1 + \alpha_2)$, $\phi(\beta_1) = \frac{1}{2}(\beta_1 - \beta_2)$, $\phi(\beta_2) = (\beta_1 + \beta_2)$.

Moreover, the associated quiver Γ is then



Of course, there is a surjective group homomorphism $f : \pi_1(Q, I) \rightarrow \pi_1(Q, I_4)$. However, there is no path from \sim_I to \sim_{I_4} in Γ .

References

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